5.1 Introduction to FDM

The finite difference techniques are based upon approximations which permit replacing differential equations by finite difference equations. These finite difference approximations are algebraic in form; they relate the value of the dependent variable at a point in the solution region to the values at some neighboring points. Thus a finite difference solution basically involves three steps:

- Dividing the solution region into a grid of nodes.
- Approximating the given differential equation by finite difference equivalent that relates the dependent variable at a point in the solution region to its values at the neighboring points.
- Solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

The course of action taken in three steps is dictated by the nature of the problem being solved, the solution region, and the boundary conditions. The most commonly used grid patterns for two-dimensional problems are shown in Fig. 5.1.

![Common grid patterns](image)

**Fig. 5.1 Common grid patterns**
(a) Rectangular grid, (b) skew grid, (c) triangular grid, and (d) circular grid.
5.2 Finite Element Schemes

Before finding the finite difference solutions to specific PDEs, we will look at how one constructs finite difference approximations from a given differential equation. This essentially involves estimating derivatives numerically.

Consider a function $f(x)$ shown in Fig. 5.2, we can approximate its derivative, slope or the tangent at $P$ by the slope of the arc $PB$, given the forward-difference formula,

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$  \hspace{1cm} (5.1)

or the slope of the arc $AP$, yielding the backward-difference formula,

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}$$  \hspace{1cm} (5.2)

or the slope of arc $AB$, resulting in the central-difference formula,

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}$$  \hspace{1cm} (5.3)

We can also estimate the second derivative of $f(x)$ at $P$ as

$$f''(x_0) \approx \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{(\Delta x)^2}$$  \hspace{1cm} (5.4)

Any approximation of a derivative in terms of values at a discrete set of points is called finite difference approximation. A more general approach is using Taylor’s series. According to the well-known expansion,
\[ f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{1}{2!}(\Delta x)^2 f''(x_0) + \frac{1}{3!}(\Delta x)^3 f'''(x_0) + \ldots \]  
(5.5)

and

\[ f(x_0 - \Delta x) = f(x_0) - \Delta x f'(x_0) + \frac{1}{2!}(\Delta x)^2 f''(x_0) - \frac{1}{3!}(\Delta x)^3 f'''(x_0) + \ldots \]  
(5.6)

upon adding these expansions,

\[ f(x_0 + \Delta x) + f(x_0 - \Delta x) = 2f(x_0) + (\Delta x)^2 f''(x_0) + O(\Delta x)^4 \]  
(5.7)

where \( O(\Delta x)^4 \) is the error introduced by truncating the series. Assuming that these terms are negligible, we can obtain

\[ f''(x_0) \approx \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{\Delta x^2} \]

which is Eq. (5.4). Subtracting Eq. (5.6) from Eq. (5.5) and neglecting terms of the order \( (\Delta x)^3 \) yields

\[ f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} \]

which is Eq.(5.3). This shows that the leading errors in Eqs. (5.3) and (5.4) are of the order \( (\Delta x)^2 \). Higher order finite difference approximations can be obtained by taking more terms in Taylor series expansion.

**Example 5.1.**

To apply the difference method to find the solution of a function \( \Phi(x,t) \), we divide the solution region in the x-t plane into equal rectangles or meshes of sides \( \Delta x \) and \( \Delta t \) as in Fig.5.3. We let the coordinates \((x,t)\) of a typical grid point or node be

\[
x = i\Delta x, \; i=0,1,2, \ldots
\]
\[
t = j\Delta t, \; j=0,1,2,\ldots
\]

and the value of \( \Phi \) at P be

\[ \Phi_P = \Phi(i\Delta x, j\Delta t) = \Phi(i, j) \]  
(5.8)

With this notation, the central difference approximations of the derivatives of \( \Phi \) at the \((i,j)\)th node are
\( \Phi_{x}/_{i,j} = \frac{\Phi(i+1,j) - \Phi(i-1,j)}{2\Delta x}, \) \hspace{1cm} (5.9a)

\( \Phi_{t}/_{i,j} = \frac{\Phi(i,j+1) - \Phi(i,j-1)}{2\Delta t}, \) \hspace{1cm} (5.9b)

\( \Phi_{xx}/_{i,j} = \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta x)^2}, \) \hspace{1cm} (5.9c)

\( \Phi_{tt}/_{i,j} = \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{(\Delta t)^2}, \) \hspace{1cm} (5.9d)

Fig. 5.3 FD mesh for two independent variables \( x \) and \( t \).

**Example 5.2**

Laplace’s equation in two dimensions may then be expressed as

\[ \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = \frac{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 - 4\Phi_0}{h^2} \] \hspace{1cm} (5.10)

Eq. (5.10) is known as the five-point equal arm difference equation using Fig. 5.4.

The FD solution procedure of Poisson’s or Laplace’s equations may then be summarized as follows:
- Divide the domain of interest (in which the potential is to be determined) into suitable fine grid. Instead of a solution for \( \Phi(x,y) \), which provides its continuous variation for a given charge distribution \( \rho(x,y) \), the FD solution will provide discrete values of \( \Phi \) at the “nodes” of the established grid.
• Apply the difference equation at each node of the grid to obtain, for example, N equations in the N unknown node potentials.
• Solve the resulting system of equations, either iteratively or using one of the direct methods.

![Five-point star grid](image1)

**Fig. 5.4 Geometry of the five-point star used in 2-D difference equations.**

### 5.3 Difference Equation at Interface between Two Dielectric Media

In many engineering applications, interfaces between two different dielectric media are encountered. For this, we will derive a special case difference equation that should be satisfied at nodes on the interface between two dielectrics. Fig. 5.5 illustrates the geometry of an interface, and the difference equation in this case may be obtained from Gauss’s law for the electric field,

![Grid nodes interface](image2)

**Fig. 5.5 Geometry of grid nodes at the interface between medium 1 of $\varepsilon_1$ and medium 2 of $\varepsilon_2$.**
\[ \oint_S \varepsilon \mathbf{E} \cdot ds = q = 0 \]  

(5.11)

$q = 0$ in eq. (5.11) because there is no “free” charge enclosed by the surface $s$. Substituting $E = -\nabla \Phi$, we obtain

\[ \oint_S \varepsilon \nabla \Phi \cdot ds = \oint_c \varepsilon \nabla \Phi \cdot dc = \oint_c \varepsilon \frac{\partial \Phi}{\partial n} dc \]  

(5.12)

The surface integration on the left-hand side of equation (5.12) was replaced by a contour integration, because in Fig. 5.5 we are dealing with a two-dimensional case, and the solution of $\Phi$ is independent of the axial independent variable $z$. $\partial \Phi / \partial n$ denotes the normal derivative of $\Phi$ on the contour $c$.

Carrying out detailed integration of $\partial \Phi / \partial n$ along $c$, we obtain

\[ \oint_c \frac{\partial \Phi}{\partial n} (\omega dc) = \frac{\Phi_1 - \Phi_0}{h} \left( \varepsilon_2 \frac{h}{2} + \varepsilon_1 \frac{h}{2} \right) + \frac{\Phi_3 - \Phi_0}{h} (\varepsilon_1 h) + \frac{\Phi_2 - \Phi_0}{h} \left( \varepsilon_1 \frac{h}{2} + \varepsilon_2 \frac{h}{2} \right) + \frac{\Phi_4 - \Phi_0}{h} (\varepsilon_2 h) \]  

(5.13)

Rearranging the terms in equation (5.13), we obtain

\[ 2\varepsilon_1 \Phi_3 + 2\varepsilon_2 \Phi_4 + (\varepsilon_1 + \varepsilon_2)\Phi_1 + (\varepsilon_1 + \varepsilon_2)\Phi_2 - 4(\varepsilon_1 + \varepsilon_2)\Phi_0 = 0 \]  

(5.14)

The equation (5.14) may be used at interfaces between two dielectrics. Its use in engineering problems will be illustrated by following examples.

5.4 FDM for PDEs

5.4.1 FD of Diffusion PDEs

\[ k \frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2} \]  

(5.15)

where $k$ is a constant. The equivalent FD approximation is

\[ k \frac{\Phi(i, j + 1) - \Phi(i, j)}{\Delta t} = \frac{\Phi(i + 1, j) - 2\Phi(i, j) + \Phi(i - 1, j)}{(\Delta x)^2} \]  

(5.16)

where $x = i\Delta x$, $i = 0, 1, 2, ..., n$, $t = j\Delta t$, $j = 0, 1, 2, ...$. In Eq. (5.16), we have used the forward difference formula for the derivative with respect to $t$ and central difference formula for that with respect to $x$. If we let
\[
r = \frac{\Delta t}{k(\Delta x)^2}
\]
(5.17)

then Eq. (5.16) can be written as
\[
\Phi(i, j + 1) = r\Phi(i + 1, j) + (1 - 2r)\Phi(i, j) + r\Phi(i - 1, j)
\]
(5.18)

This explicit formula can be used to compute \(\Phi(x, t + \Delta t)\) explicitly in terms of \(\Phi(x, t)\).

Thus the values of \(\Phi\) along the first time row (see Fig. 5.3), \(t = \Delta t\), can be calculated in terms of the boundary and initial conditions, then the values of \(\Phi\) along the second time row, \(t = 2\Delta t\), are calculated in terms of the first time row, and so on.

In order to ensure a stable solution or reduce errors, care must be exercised in selecting the value of \(r\) in Eq. (5.17) and Eq.(5.18) is valid only if the coefficient \((1-2r)\) in Eq.(5.18) is nonnegative or \(0 < r \leq 1/2\). If we choose \(r = 1/2\), Eq. (5.18) becomes
\[
\Phi(i, j + 1) = \frac{1}{2}[\Phi(i + 1, j) + \Phi(i - 1, j)]
\]
(5.19)

5.4.2 FD of Helmholtz Equation
\[
\nabla^2 \Phi + k^2 \Phi = 0
\]
(5.20)

where \(\Phi = E_z\) for TM modes or \(\Phi = H_z\) for TE modes in waveguide problems, while \(k\) is the wave number given by \(k^2 = \omega^2 \mu \epsilon - \beta^2\).

To apply the FD method, we discretize the cross section of the waveguide by a suitable mesh. Applying central difference approximation for partial derivatives to Eq. (5.20) gives
\[
\Phi(i + 1, j) + \Phi(i - 1, j) + \Phi(i, j + 1) + \Phi(i, j - 1) - (4 - h^2 k^2)\Phi(i, j) = 0
\]
(5.21)

5.4.3 FD of Wave Equation
\[
u^2 \frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial t^2}
\]
(5.22)

where \(u\) is the speed of the wave. An equivalent FD formula is
\[
u^2 \frac{\Phi(i + 1, j) - \Phi(i, j) + \Phi(i - 1, j)}{(\Delta x)^2} = \frac{\Phi(i, j + 1) - 2\Phi(i, j) + \Phi(i, j - 1)}{(\Delta t)^2},
\]
where \(x = i\Delta x, i = 0,1,2,..., n, t = j\Delta t, j = 0,1,2,...\). This equation can be written as
\[ \Phi(i, j + 1) = 2(1 - r)\Phi(i, j) + r[\Phi(i + 1, j) + \Phi(i - 1, j)] - \Phi(i, j - 1) \quad (5.23) \]

where \( \Phi(i, j + 1) \) is an approximation to \( \Phi(x,t) \) and \( r \) is the “aspect ratio” given by

\[ r = \left( \frac{u\Delta t}{\Delta x} \right)^2 \quad (5.24) \]

Equation (5.23) is an explicit formula for the wave equation. For the solution algorithm in Eq.(5.23) to be stable, the aspect ratio \( r < 1 \) will be chosen. If we choose \( r = 1 \), Eq. (5.23) becomes

\[ \Phi(i, j + 1) = \Phi(i + 1, j) + \Phi(i - 1, j) - \Phi(i, j - 1) \quad (5.25) \]

Unlike the single-step schemes of Eqs.(5.18), the two-step schemes of Eqs. (5.23) and (5.25) require that the values of \( \Phi \) at times \( j \) and \( j - 1 \) be known to get \( \Phi \) at time \( j + 1 \). Thus, we must derive a separate algorithm to “start” the solution of Eq. (5.23) or (5.25); that is, we must compute \( \Phi(i,1) \) and \( \Phi(i,2) \). To do this, we utilize the prescribed initial condition. For example, suppose the initial condition on the FDE in Eq. (5.22) is

\[ \frac{\partial \Phi}{\partial t}(x,0) = 0 \]

We use the backward-difference formula

\[ \frac{\partial \Phi(x,0)}{\partial t} = \Phi(i,1) - \Phi(i,-1) = 0 \]

or

\[ \Phi(i,1) = \Phi(i,-1) \quad (5.26) \]

Substituting Eq. (5.26) into Eq. (5.23) and taking \( j = 0 \) (i.e., at \( t = 0 \)), we can obtain

\[ \Phi(i,1) = (1 - r)\Phi(i,0) + \frac{r}{2} [\Phi(i - 1,0) + \Phi(i + 1,0)] \quad (5.27) \]

Using the starting formula Eq. (5.27) together with the prescribed boundary and initial conditions, the value of \( \Phi \) at any grid point \( (i,j) \) can be obtained directly from Eq. (5.23).

**Accuracy and Stability of FD Solutions**

There are three sources of errors that are nearly unavoidable in numerical solution of physical problems:

1. modeling errors,
2. truncation (or discretization) errors,
3. round-off errors.
Each of these error types will affect accuracy and therefore degrade the solution. Fig. 5.6 shows us the error as a function of the mesh size.

![Graph showing error as a function of mesh size]

Fig. 5.6 Error as a function of the mesh size

5.5 Example of 2D FDM

Consider the rectangular region shown in Fig. 5.7a. The electric potential is specified on the conducting boundaries. Use the finite difference representation to solve for the potential distribution within this region.

![Diagram of rectangular geometry with electric potential boundary conditions]

Fig. 5.7a Rectangular geometry and boundary condition for the electric potential problem.
**Solution**

The electric potential everywhere in the rectangular region should satisfy Laplace’s equation. Using a numerical solution means we will define $\Phi$ in the rectangular region of interest by calculating its values at discrete points, the nodes of a mesh. The step-by-step solution procedure includes the following:

1. Layout a coarse square mesh and identify the nodes at which the electric potential is to be calculated. The geometry of a $2 \times 4$ mesh is shown in Fig. 5.7b. The value of $h$ (mesh size) in this case is $h = 5\text{cm}$.

2. Replace Laplace’s equation by its finite difference representation.

$\frac{\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j}}{h^2} = 0$ \hspace{1cm} (5.28)

$\Phi_{i,j}$ are the discrete values of the potential at points (nodes) within the domain of interest.

3. Apply the difference equation in step 2 at each node. At node 1,

$\frac{0 + 0 + 0 + \Phi_2 - 4\Phi_1}{(0.05)^2} = 0$

or

$4\Phi_1 = \Phi_2$ \hspace{1cm} (5.29)

At node 2,

$\frac{\Phi_1 + \Phi_3 + 0 + 0 - 4\Phi_2}{(0.05)^2} = 0$
or 
\[ \Phi_1 - 4\Phi_2 + \Phi_3 = 0 \] 
(5.30)

At node 3, 
\[ \frac{\Phi_2 + 100 - 4\Phi_3}{(0.05)^2} = 0 \]

or 
\[ \Phi_2 - 4\Phi_3 = -100 \] 
(5.31)

(4) Eqs. (5.29) to (5.31) are three equations in the three unknowns, \(\Phi_1, \Phi_2\) and \(\Phi_3\). These three equations may be solved using one of the methods described in calculus courses. The results are 
\[ \Phi_1 = 1.79, \quad \Phi_2 = 7.14, \quad \Phi_3 = 26.79 \]

(5) With the coarse mesh we used, we do not expect to get accurate final results. Redoing the problem with a smaller value of \(h\) should improve the accuracy of the solution. Fig. 5.7c shows the mesh geometry for \(h = 2.5\) cm, which is half the mesh size used in the previous calculations. In this case, however, we have twenty-one unknown values of the potential at the various nodes.

(6) Once again, applying the difference equation of step 2 at the various nodes results in the following 21 x 21 matrix:
Table 5.1 Comparison between Finite Difference and Analytical Results

<table>
<thead>
<tr>
<th></th>
<th>Potential values</th>
<th>Percentage error</th>
<th>Potential values</th>
<th>Percentage error</th>
<th>Analytical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_9$</td>
<td>$1.786$</td>
<td>$63$</td>
<td>$1.289$</td>
<td>$17.80$</td>
<td>$1.094$</td>
</tr>
<tr>
<td>$\Phi_{11}$</td>
<td>$7.143$</td>
<td>$30$</td>
<td>$6.019$</td>
<td>$9.70$</td>
<td>$5.489$</td>
</tr>
<tr>
<td>$\Phi_{13}$</td>
<td>$26.786$</td>
<td>$2.7$</td>
<td>$26.289$</td>
<td>$0.75$</td>
<td>$26.094$</td>
</tr>
</tbody>
</table>

Table 5.1 compares the results for $h = 5$ cm, and $h = 2.5$ cm. From this comparison, it is clear that the $h = 2.5$ cm results agree better with the analytical solution available for this simple geometry. As expected, the accuracy of the FD results improves with the reduction in size $h$. Clearly, any further reduction in $h$ results in a larger-size matrix; hence, a compromise should be made between the desired accuracy and the computational time and effort required.

The solution for the electric potential at the various nodes is given by

$\Phi_1 = 0.353$  $\Phi_8 = 0.499$  $\Phi_{15} = 0.353$
$\Phi_2 = 0.913$  $\Phi_9 = 1.289$  $\Phi_{16} = 0.913$
$\Phi_3 = 2.010$  $\Phi_{10} = 2.832$ $\Phi_{17} = 2.010$
$\Phi_4 = 4.296$  $\Phi_{11} = 6.019$ $\Phi_{18} = 4.296$
$\Phi_5 = 9.153$  $\Phi_{12} = 12.654$ $\Phi_{19} = 9.153$
$\Phi_6 = 19.663$ $\Phi_{13} = 26.289$ $\Phi_{20} = 19.663$
$\Phi_7 = 43.210$ $\Phi_{14} = 53.177$ $\Phi_{21} = 43.210$
EXAMPLE 2

In the $6 \times 8m^2$ rectangular region shown in Figure 2.2(a), the electric potential is zero on the boundaries. The charge distribution, however, is uniform and given by $p_v = 2\varepsilon_0$.

Solve Poisson’s equation to determine the potential distribution in the rectangular region.

Solution

To determine the potential distribution in the rectangular region, we use Poisson’s equation.

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -\frac{\rho}{\varepsilon_0} = -2$$

with zero potential $\Phi = 0$ on the boundaries.

By establishing the rectangular grid shown in Figure 2.2(b), we realize that we have six nodes and, hence, six unknown potentials for which to solve. Replacing $\nabla^2 \Phi$ by its finite difference representation, we obtain

$$\frac{1}{h^2} (\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j}) + 2 = 0$$
It should be noted that although the mesh size was not explicitly used in solving Laplace’s equation in the previous example, \( h \) is included as a part of the matrix formation in solving Poisson’s equation. In SI system of units, \( h \) should be in meters. By applying the preceding difference equation at the various nodes in Figure 2.2(b), we obtain the following matrix equation:

\[
\begin{bmatrix}
-4 & 1 & 1 & 0 & 0 & 0 \\
1 & -4 & 0 & 1 & 0 & 0 \\
1 & 0 & -4 & 1 & 1 & 0 \\
0 & 1 & 1 & -4 & 0 & 1 \\
0 & 0 & 1 & 0 & -4 & 1 \\
0 & 0 & 0 & 1 & 1 & -4 \\
\end{bmatrix}
\begin{bmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3 \\
\Phi_4 \\
\Phi_5 \\
\Phi_6 \\
\end{bmatrix}
= 
\begin{bmatrix}
-8 \\
-8 \\
-8 \\
-8 \\
-8 \\
-8 \\
\end{bmatrix}
\]

Instead of solving the resulting six equations, we may note some symmetry considerations in Figure 2.2(b). It is clear that

\[
\Phi_1 = \Phi_2 = \Phi_5 = \Phi_6 \quad \text{and that} \quad \Phi_3 = \Phi_4
\]

Taking these symmetry considerations into account, the number of equations reduces to two, and we obtain the following solution:

\[
\Phi_1 = 4.56, \quad \Phi_3 = 5.72
\]

To improve the accuracy of the potential distribution, finer mesh such as the one shown in Figure 2.2 is required. Because of the large number of nodes in this case, symmetry should be used, and a solution for only one-quarter of the rectangular geometry is desired. The application of the difference equation at nodes 1, 2, 4, and 5 should proceed routinely, whereas special care should be exercised at the boundary nodes 3, 6, 9, 10, and 11, and also at the corner node 12.

For example, applying the difference equation at node 6 yields

\[
\frac{1}{h^2} (\Phi_a + \Phi_3 + \Phi_9 + \Phi_5 - 4\Phi_6) + 2 = 0
\]

Or

\[
\frac{1}{h^2} (\Phi_3 + \Phi_9 + 2\Phi_5 - 4\Phi_6) + 2 = 0
\]
In equation \( \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{1}{h^2} (\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 - 4\Phi_6) \), symmetry was used to complete the five-point star difference equation. Specifically the potential at node a to the right of 6 was taken equal to \( \Phi_5 \). Similarly at the corner node 12, we obtain

\[
\frac{1}{h^2} (\Phi_{11} + \Phi_9 + \Phi_{b} + \Phi_{c} - 4\Phi_{12}) + 2 = 0
\]

Because of symmetry, \( \Phi_{11} = \Phi_{b} \) and \( \Phi_9 = \Phi_{c} \), hence,

\[
\frac{1}{h^2} (2\Phi_{11} + 2\Phi_9 - 4\Phi_{12}) + 2 = 0
\]
The “2” coefficient in the coefficient matrix (to the left) of equation appears whenever symmetry consideration is used at boundary and corner nodes. It should be noted that the 12×12 coefficient matrix in equation is the same for both Laplace’s and Poissons equations. The constant vector on the right-hand side of equation, however, depends on the charge distribution within and the potential at the boundaries of the region of interest. Furthermore, if instead of a uniform charge distribution we have a given charge distribution \( \rho(x,y) \), the constants vector on the right-hand side of equation should reflect the value of \( \rho(x,y) \) calculated at each node. Solution of equation gives

\[
\begin{bmatrix}
\Phi_1 & \Phi_2 & \Phi_3 \\
\Phi_4 & \Phi_5 & \Phi_6 \\
\Phi_7 & \Phi_8 & \Phi_9 \\
\Phi_{10} & \Phi_{11} & \Phi_{12}
\end{bmatrix}
= \begin{bmatrix}
-2 \\
-2 \\
-2 \\
-2
\end{bmatrix}
\]

\[
\begin{bmatrix}
-4 & 1 & 1 \\
1 & -4 & 1 \\
2 & -4 & 1 \\
1 & 1 & -4 \\
1 & 2 & -4 \\
1 & 1 & -4 \\
1 & 2 & -4 \\
2 & 1 & -4 \\
2 & 2 & -4
\end{bmatrix}
\]

\[
\Phi_1 = 2.04, \quad \Phi_2 = 3.05, \quad \Phi_3 = 3.35 \\
\Phi_4 = 3.12, \quad \Phi_5 = 4.79, \quad \Phi_6 = 5.32 \\
\Phi_7 = 3.66, \quad \Phi_8 = 5.69, \quad \Phi_9 = 6.34 \\
\Phi_{10} = 3.82, \quad \Phi_{11} = 5.96, \quad \Phi_{12} = 6.65
\]