Conductivity and Resistivity Tensor Rotation for Anisotropic Surface Impedance Modelling

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Abstract – The electromagnetic surface impedance of a half-space with dipping conductivity anisotropy can be derived from the isotropic half-space solution provided the conductivity used in the expressions are for the effective horizontal conductivity. As a solution to Maxwell’s equations for a TM-mode plane wave incidence, the effective horizontal conductivity must be derived from the tensor rotation of the resistivity tensor and not the tensor rotation of the conductivity tensor.

I. FORMULATION

At very low frequencies for most earth environments, conduction currents dominate displacement currents and Maxwell’s equations in differential form for anisotropic media can be written as:

\[
\nabla \times \mathbf{E} = -j\omega \mathbf{H} \quad (1) \\
\nabla \times \mathbf{H} = \sigma \mathbf{E} \quad (2)
\]

where fields of time variance \( e^{j\omega t} \) are implied. It is necessary to solve Maxwell’s equations for the anisotropic half-space region \( z' < 0 \) for an isotropic \( \mu \) and anisotropic \( \sigma \), which is a two-dimensional tensor expressed as:

\[
\sigma = \begin{pmatrix} \sigma_t & 0 \\ 0 & \sigma_n \end{pmatrix} \quad (3)
\]

where \( t \) subscripts indicate the conductivity tangential to the surface and \( n \) subscripts indicate the conductivity normal to the surface. \( \alpha \) is the inclination of the anisotropy planes (Fig. 1). Following [1], for a TM mode of propagation incident transverse to the strike of the anisotropy, Maxwell’s equations then take the form:

\[
\frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} + j\omega \mu H_x = 0 \quad (4)
\]

\[
\sigma_t \frac{\partial E_y}{\partial y} + \sigma_n \frac{\partial E_z}{\partial z} = 0 \quad (5)
\]

\[
E_y = \frac{1}{\sigma_t} \frac{\partial H_x}{\partial z} \quad (6)
\]

\[
E_z = -\frac{1}{\sigma_n} \frac{\partial H_x}{\partial y} \quad (7)
\]

For sufficiently low frequencies, displacement currents are neglected in both the air and the half-space so the problem reduces to the solution of the field equations in the conducting half-space only. The boundary conditions imposed on the problem are that the fields attenuate to zero for \( z' \to \infty \) and the normal component of the current density \( J_z \) is equal to zero at the surface, \( z' = 0 \).

Figure 1. Half-space with two-dimensional anisotropy inclined at an arbitrary angle \( \alpha \).

From the above boundary conditions and (4) – (7), then:

\[
\frac{1}{\sigma_n} \frac{\partial^2 H_x}{\partial y^2} + \frac{1}{\sigma_t} \frac{\partial^2 H_x}{\partial z^2} - j\omega \mu H_x = 0 \quad (8)
\]

for component \( H_x \) which has a solution in the form of a plane wave:

\[
H_x = Ce^{-qz'} = Ce^{-q(y \sin \alpha + z \cos \alpha)} \quad (9)
\]

which satisfies (8) when the wave number \( q \) is given by:

\[
q = \sqrt{\frac{j\omega \mu \sigma}{1 + (K^2 - 1) \sin^2 \alpha}} \quad (10)
\]

where \( K = \frac{\sigma_n}{\sigma_t} \geq 1 \). In order that the wave be attenuated as the depth increases, \( q \) must be chosen such that \( \text{Re } q < 0 \). \( C \) is an arbitrary constant and is real number. From Maxwell’s equations (6)-(7) respectively, expressions for the electric field components are then given as:

\[
E_y = \frac{-q \cos \alpha}{\sigma_t} Ce^{-q(y \sin \alpha + z \cos \alpha)} \quad (11)
\]

\[
E_z = \frac{-q \cos \alpha}{\sigma_n} Ce^{-q(y \sin \alpha + z \cos \alpha)}
\]
\[ E_z = \frac{q \sin \alpha}{\sigma_n} C e^{q(y \sin \alpha + z \cos \alpha)} \] (12)

However, in order to apply the boundary conditions, an expression for the horizontal component of the electric field must be obtained. Using co-ordinate rotation:

\[ E_y' = E_y \cos \alpha - E_z \sin \alpha \] (13)

and substituting (11) and (12), the horizontal electric field is given by:

\[ E_y' = q C e^{q(y \sin \alpha + z \cos \alpha)} \left( \frac{\cos^2 \alpha}{\sigma_t} + \frac{\sin^2 \alpha}{\sigma_n} \right), \]

by observation which can be reduced to:

\[ E_y' = q \left( \frac{\cos^2 \alpha}{\sigma_t} + \frac{\sin^2 \alpha}{\sigma_n} \right) H_x \] (14)

The surface impedance is then defined as:

\[ Z_s = \frac{E_y'}{H_x} = q \left( \frac{\cos^2 \alpha}{\sigma_t} + \frac{\sin^2 \alpha}{\sigma_n} \right) \] (15)

which can be reduced to:

\[ Z_s = \sqrt{\frac{j \omega \mu}{\sigma_t} \{ 1 + (K^2 - 1) \sin^2 \alpha \}} \] (16)

using (10). When \( \alpha = 0^\circ \) or \( 90^\circ \), (16) reduces to an isotropic half-space problem where:

\[ Z_s = \sqrt{\frac{j \omega \mu}{\sigma_t}} \] (17)

by definition, for a half-space with inclined anisotropy, the effective horizontal conductivity is given by:

\[ \sigma = \left( \frac{\cos^2 \alpha}{\sigma_t} + \frac{\sin^2 \alpha}{\sigma_n} \right)^{-1} \] (17)

which is the reciprocal of the pre- and post-multiplication tensor rotation of the resistivity tensor into the horizontal plane:

\[ \rho' = \rho A^T \] (18)

where \( \rho = \begin{pmatrix} \rho_t & 0 \\ 0 & \rho_n \end{pmatrix} \), \( \rho_t = \sigma_t^{-1} \), \( \rho_n = \sigma_n^{-1} \),

\[ A = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \] and \( A^T \) is the transpose of \( A \). It is to this point that we critically define the effective horizontal conductivity by (17) and not the expression:

\[ \sigma = \sigma_t \cos^2 \alpha + \sigma_n \sin^2 \alpha \] (19)

which is the pre- and post-multiplication tensor rotation of the conductivity tensor into the horizontal plane. (19) is not a solution to Maxwell’s equations in the case presented here [2]. This important distinction can be easily extended to calculate the surface impedance of a horizontally stratified half-space with each layer having inclined anisotropies [3], [4] by using the effective horizontal conductivity in the isotropic model [5].

III. CONCLUSIONS

It has been demonstrated that the effective horizontal conductivity for a half-space with inclined anisotropy must be calculated from the tensor rotation of the resistivity tensor and not from the tensor rotation of the conductivity tensor.

REFERENCES


