TABLE V
TAP LOCATION AND COEFFICIENTS, EXAMPLE IV, $N = 7$ ($N_0 = 13$)

Taps :	0	1.391	2.782	5.566	6.959	9.753	10.904
Coeff:	.3332	.5430	.2596	-.1074	-.0741	.0331	.0237

sical approaches. This is expected because the nonequidistant-tap design is conceptually equivalent to a procedure of skipping the slightly weighted taps and moving the others to anticipate for that omission. The small coefficient range is a very desirable feature in implementation where coarsely quantized coefficients have to be used.

IV. DISCUSSION

Desired filter responses can be approximated by nonrecursive structures more economically when the taps are allowed to take suitable positions, spaced, in general, unequally on the delay line. On the other hand, digital implementation of such a line with nonequidistant taps is feasible if digitizing techniques with inherent high clock rates, like delta modulation, are used.

In this paper, results were presented on FIR filter design with nonequidistant taps. The design method guarantees local optimality, while the derived responses are close to their counterparts of the equidistant-tap design with the same total delay-line length but with many more taps.

A useful observation on the examples considered is that the coefficient dynamic range is much narrower now, which is a very desirable feature in cases of a coarse quantization of the coefficients.

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Further Simulation Results on Tapered and Energy-Weighted Burg Methods

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Abstract—**In this correspondence, we report about a simulation study of the tapered and energy-weighted Burg methods for sinusoidal signals. We confirm Swingler's results about the frequency estimation error and extend them for frequency resolution. We also discuss some drawbacks of the energy-weighted Burg methods. On the basis of the results reported in this correspondence, we show that the optimum-tapered Burg method results in the best performance.**

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I. INTRODUCTION

The Burg method of autoregressive (AR) spectral estimation is known to result in relatively high values of phase-dependent bias (or frequency estimation error) when applied to short data records of the sinusoidal signals [1]. Recently, some modifications over the Burg method have been proposed [2]–[5] which try to solve this problem. These modifications use some form of weighting of the forward and backward prediction error energies prior to their minimization under the constraint of Levinson's recursion. For example, Swingler [2] has used the Hamming weighting, Kaveh and Lippert [3] the optimum tapered weighting, and Scott and Nikias [4] the energy weighting. By observing the success of the tapered and energy-weighted Burg methods [2]–[4], we have suggested earlier the use of a window function which is a product of optimum-tapering and energy-weighting window functions [5].

Recently, Swingler [6] has studied the performance of the tapered Burg methods in terms of frequency estimation error (using an analytical expression [7]) and has shown that the Hamming-windowed Burg method performs better than the optimum-tapered Burg method in the midfrequency $[2\pi/N, \pi - 2\pi/N]$ (where N is the number of data points of the sinusoidal signal being analyzed). Unfortunately, Swingler has not addressed the following four interesting points in this study [6]. First, since the analytical expression for the frequency estimation error is not valid in the frequency regions near 0 and π, his study could not comment about the performance of the tapered Burg methods in these frequency regions. Second, he has not studied the performance of these methods in terms of frequency resolution. Third, he has not included the energy-weighted Burg methods in his study. Fourth, he has studied the performance of the tapered Burg methods for noise-free sinusoidal signals only. It might be interesting to study these methods for the sinusoidal signals degraded by the addition of white Gaussian noise. The aim of the present correspondence is to answer these points through a simulation study and to supplement Swingler's results.

II. SIMULATION RESULTS

We report here the simulation results for the original Burg method (BURG), the Hamming-windowed Burg method (BURGH), the optimum-tapered Burg method (BURGO), the energy-weighted Burg method (BURGE), and the product-weighted Burg method (BURGP). In the following, $f\ (=\theta/2\pi)$ is the frequency (in hertz) of the sinusoidal signal, ϕ is the initial phase (in degrees), A is the amplitude of the sinusoid, f_s is the sampling frequency (in hertz), N is the number of data samples, M is the AR model order used for spectral estimation, and σ^2 is the variance of zero-mean white Gaussian noise added to the sinusoid. We define the signal-to-noise ratio in decibels as $\mathrm{SNR} = 10 \log_{10} (A^2/2\sigma^2)$.

We first employ the example of noise-free single-sinusoidal signal used by Swingler [6] (i.e., $A = 1, f_s = 1$ Hz, $N = 16$, $M = 2$, and $\sigma^2 = 0$). Here we vary the initial phase ϕ from 0 to 2π and estimate the sinusoid's frequency and bandwidth (BW) by solving the second-order polynomial representing linear prediction error filter for its roots [8, eq. (7.1), p. 167]. (Following Nikias and Scott [9], we define the estimated BW as an index of resolution degradation for the single-sinusoidal signal.) The quantities Δf^2 and BW are averaged over the phase ϕ to compute the frequency error variance $\langle \Delta f^2 \rangle_\phi$ and the average bandwidth $\langle \mathrm{BW} \rangle_\phi$, respectively. These average quantities, $\langle \Delta f^2 \rangle_\phi$ and $\langle \mathrm{BW} \rangle_\phi$, are then computed for different values of the sinusoid's frequency θ in the range of 0 to π.

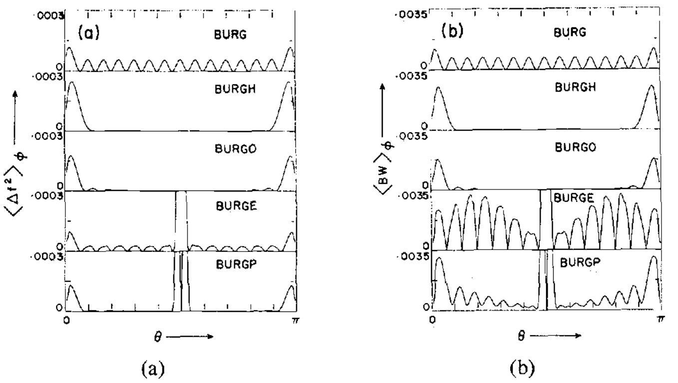

Fig. 1. (a) Frequency error variance and (b) average bandwidth as a function of sinusoid's frequency for the noise-free single-sinusoidal signal.

TABLE I
$\langle \Delta f^2 \rangle_{\phi,\theta}$ AND $\langle \mathrm{BW} \rangle_{\phi,\theta}$ FOR NOISE-FREE SINGLE-SINUSOIDAL SIGNAL

Method	$\langle \Delta f^2 \rangle_{\phi,\theta}$ (Hz^2)	$\langle \mathrm{BW} \rangle_{\phi,\theta}$ (Hz)
BURG	0.000033	0.00039
BURGH	0.000029	0.00029
BURGO	0.000018	0.00018
BURGE	0.000156	0.00302
BURGP	0.000206	0.00245

Variation of $\langle \Delta f^2 \rangle_\phi$ and $\langle \mathrm{BW} \rangle_\phi$ with respect to frequency θ is shown in Fig. 1. We also compute frequency-averaged values of the frequency error variance and average bandwidth (i.e., $\langle \Delta f^2 \rangle_{\phi,\theta}$ and $\langle \mathrm{BW} \rangle_{\phi,\theta}$). These are listed in Table I. We can make the following observations from Fig. 1 and Table I.

1) It can be seen from Table I that the tapered Burg methods (BURGH and BURGO) result in smaller values of $\langle \Delta f^2 \rangle_{\phi,\theta}$ and $\langle \mathrm{BW} \rangle_{\phi,\theta}$ than the BURG method, while the BURGE and BURGP methods result in larger values. Among the tapered Burg methods, the BURGO method leads to smaller values of $\langle \Delta f^2 \rangle_{\phi,\theta}$ and $\langle \mathrm{BW} \rangle_{\phi,\theta}$ than the BURGH method.

2) We can see from Fig. 1(a) that in the frequency regions near 0 and π, the BURGO method results in smaller values of $\langle \Delta f^2 \rangle_\phi$ than the BURGH method. However, if we exclude these regions (i.e., if we consider the midfrequency region $[2\pi/N, \pi - 2\pi/N]$), we find that $\langle \Delta f^2 \rangle_\phi$ resulting from the BURGH method is marginally smaller than that from the BURGO method. We see similar results for the tapered Burg methods from Fig. 1(b) for $\langle \mathrm{BW} \rangle_\phi$. Thus, we have confirmed, through simulation, Swingler's results for $\langle \Delta f^2 \rangle_\phi$ for the midfrequency region and we have extended them to show similar results for $\langle \mathrm{BW} \rangle_\phi$. In addition, we have commented about the behavior of $\langle \Delta f^2 \rangle_\phi$ and $\langle \mathrm{BW} \rangle_\phi$ for frequencies in the regions near 0 and π.

3) The BURGE and BURGP methods show large values of $\langle \Delta f^2 \rangle_\phi$ and $\langle \mathrm{BW} \rangle_\phi$ for frequencies in the region near $\theta = \pi/2$ which is not seen for other methods. If we exclude frequency regions near 0, $\pi/2$, and π, we can see from Fig. 1(a) that the BURGE method results in smaller values of $\langle \Delta f^2 \rangle_\phi$ than the BURG method (as expected from the analytical results derived by Nikias and Scott [10]), but larger values than the BURGH, BURGO, and BURGP methods. The BURGP method gives slightly smaller values of $\langle \Delta f^2 \rangle_\phi$ than the BURGO method, but slightly larger values than the BURGH method. However, examination of Fig. 1(b) reveals that in terms of resolution, the BURGE method as well as the BURGP method perform poorer than the BURG, BURGH, and BURGO methods for all the frequencies.

So far, we have studied a noise-free sinusoidal signal. Now we study the sinusoidal signals of more practical interest (i.e., with additive white Gaussian noise). We first employ an example of a noisy single-sinusoidal signal with $M = 9$ and SNR $= 25$ dB. For this example, the frequency error variance is studied as a function of the sinusoid's frequency and results similar to those discussed earlier for noise-free sinusoidal signals are observed. Next, we employ an example of the two-sinusoidal signal in additive white Gaussian noise. The two sinusoid components have here SNR's of 30 and 27 dB. The estimated error spectra for the BURG, BURGH, BURGO, BURGE, and BURGP methods are shown in Fig. 2. It can be seen from this figure that the BURGO method shows the best performance among the five methods (i.e., it results in the least bias and variance in estimating the sinusoidal frequencies).

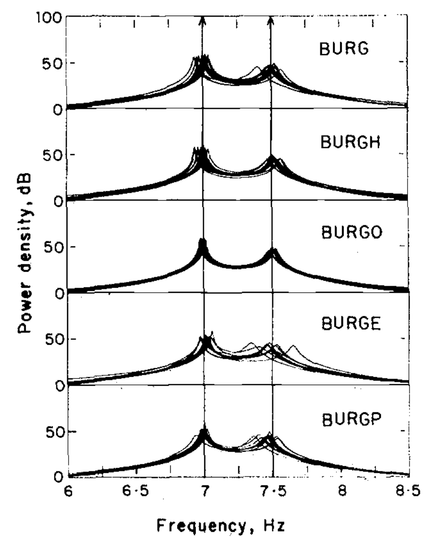

Fig. 2. Estimated power spectra for the two-sinusoidal signal ($f_1 = 7$ Hz, $A_1 = 1$, $f_2 = 7.5$ Hz, $A_2 = 0.5$, $f_s = 32$ Hz, $N = 80$, $M = 12$, $\sigma^2 = 0.0005$). Ten different noise realizations are used with random initial phases.

III. CONCLUSION

A simulation study of the tapered and energy-weighted Burg methods is reported. It is shown that the tapered Burg methods (i.e., BURGH and BURGO) improve the performance of the BURG method for the noise-free single-sinusoidal signals, while the BURGE method does not always improve it (it especially suffers from poor resolution). Since the BURGP method employs both optimum tapering and energy weighting, it inherits advantages as well as disadvantages of the BURGO and BURGE methods. It shows selective improvement over the BURG method in terms of the frequency estimation error in the midfrequency region where both the BURGO and BURGE methods work better than the BURG method. But like the BURGE method, this method also suffers from poor resolution.

It is also shown that the BURGO method gives better results than the BURGH method for the single-sinusoidal signals when we consider the total frqeuency region $[0, \pi]$. But in the midfrequency region $[2\pi/N, \pi - 2\pi/N]$, the BURGH method results in smaller frequency error variance and better resolution than the BURGO method. However, we show that these differences (occurring in the midfrequency region) resulting from the two Burg methods are very small and if we use these methods for noisy and multicomponent sinusoidal signals, the BURGO method performs better than the BURGH method.

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Improved Long Convolutions Using Generalized Number Theoretic and Polynomial Transforms

SOO-CHANG PEI AND JA-LING WU

Abstract—**Generalized number theoretic and polynomial transforms can be used in the computation of a long convolution. This will reduce the computation complexity by a factor of two; it only requires fewer than 2 real multiplications, 45 real additions, and 10 data shifts per point for a typical 2048 array.**

I. Introduction

The fast polynomial transform (FPT) for computing 2D convolutions was first introduced by Nussbaumer [1]. Truong *et al.* [2] extended the FPT concept with the Chinese remainder theorem. The generalized fast Fourier transform is used to further reduce the FPT complexity [3], [4].

Recently, Martinelli introduced a method for computing 1D long convolutions using number theoretic and polynomial transforms [5]. In this correspondence, we propose that the generalized number theoretic transform (GNTT) [6] can be used in the computation of polynomial products in the above algorithm; by suitable choice of the basis α and modulus M, the GNTT would require no multiplication for the computation of the transform. This would reduce the algorithm's complexity greatly; it is about two times faster than the current most efficient method for a typical 2048 array.

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II. Skew-Circular Convolutions Using GNTT's

The generalized number theoretic transform and its inverse (IGNTT) [6] are defined by

$$X(k) = \sum_{n=0}^{N-1} x(n)\, \alpha^{n(k+1/2)} \text{ Mod } M$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, \alpha^{-n(k+1/2)} \text{ Mod } M \qquad (1)$$

with the following conditions holding:

1) $\alpha^N = 1$ Mod M
2) $\alpha^t \neq 1$ Mod M, $\quad 0 < t < N$
3) N^{-1} exists.

Equation (1) is a special case of the generalized number theoretic transform in [6]. Thus, the algorithm in [3] and [4] can be used to compute (1) in a fast FFT-like manner. This FFT-like structure requires the same number of operations as the NTT; the premultiplications by $\alpha^{n/2}$ are nicely eliminated. If we properly choose the basis $\alpha^{1/2}$ as 2 or power of 2 in the finite field defined modulo M, then the GNTT would require no multiplication for the computation of the transform; this can be used to efficiently calculate the skew-circular convolutions of two sequences or the polynomial products modulo $(Z^N + 1)$ [6], i.e.,

$$\{x_n\} \xrightarrow{\text{GNTT}} \{X_k\}$$

$$\{h_n\} \xrightarrow{\text{GNTT}} \{H_k\}$$

$$\{x_n\} \circledast_s \{h_n\} \xleftarrow{\text{IGNTT}} \{X_k\} \cdot \{H_k\} \text{ Mod } M$$

where $\circledast_s$ denotes the skew-circular convolution operation. Using $\alpha^{1/2} = \sqrt{2}$ for the Fermat number $2^{2^b} + 1$, the maximum possible length for these GNTT's is 2^{b+2}. However, the number of additions and shifts required will be increased.

III. Complexity Measure of Improved GNTT Method for Long Convolutions

The computation of the polynomial products mod $(Z^{2^{b+1}} + 1)$ is very crucial in the complexity of the Martinelli algorithm (refer to [5, Remark 5, eq. (19)]). Martinelli computes these products by the Fermat NTT's in the extended ring $(Z^{2^{b-2}} - 1)$, and skew folding back into the $(Z^{2^{b-1}} + 1)$ ring in two steps (see [5, eq. (20) and (21)]). We propose that the GNTT's can be used to improve the above algorithm; the skew-circular convolution of two length 2^{b+1} sequences need two GNTT's (one forward and one inverse transform; the filter impulse response forward transform has been precomputed, and is not counted here) and 2^{b+1} real multiplications in the transform domain. Calculating the GNTT (and the IGNTT) requires no multiplication; this will reduce the complexity greatly by a factor of two.

For a length 2^{b+1} skew-circular convolution, we have the following.

	Multiplications	*Additions*	*Shifts*
FNT and skew-folding need	2^{b+2}	$2^{b+3}(b+2)$	$2^{b+2}(b+2)$
GNTT needs	2^{b+1}	$2^{b+2}(b+1)$	$2^{b+1}(b+1)$

For a total long length N circular convolution ($N = 2^t = N_1 \times N_2 = 2^{b+1} \times 2^{t-b-1}$), it will require $(2^{t-b} - 3)$ small skew-circular convolutions of length 2^{b+1} (see [5, eq. (22)]).