

Reconstruction of a Signal from the Real Part of Its Discrete Fourier Transform

In this tutorial, we present a procedure for reconstructing a complex-valued, discrete-time signal from only partial Fourier transform (FT) information, more specifically, the real part of its discrete FT (RDFT). By applying a delay, coupled with appropriate zero-padding to ensure a sufficiently dense sampling of the frequency axis, we show that any signal can be reconstructed perfectly from the RDFT alone. The presented procedure can, in the case of a densely sampled DFT, provide a reduction in the computational complexity of analysis-modification-synthesis-based speech processing methods that independently process the real and imaginary (RI) parts temporally. Furthermore, the perfect reconstruction property of this method implies that the RDFT alone captures all of the information about the signal, which suggests that it may be a potentially useful frequency-derived domain for the processing of speech signals.

Introduction

Discrete-time signal reconstruction from complete or partial frequency domain information is a very common task in digital signal processing. When referring to frequency domain information, we mean in terms of the magnitude and phase components, or RI parts that are computed from the DFT of the signal. By using the inverse DFT with appropriate zero-padding, a discrete-time signal

can be completely and uniquely determined, if both the magnitude and phase, or similarly, both the RI parts at each frequency sample are known.

Some previous literature has studied signal reconstruction techniques from partial FT information. Van Hove et al. [1] showed that a one- or two-dimensional signal can be uniquely specified by its FT magnitude and limited (1-bit) FT phase information. An iterative reconstruction algorithm was presented, where the mean squared error of the reconstructed sequence decreased monotonically with each iteration, provided that the “signed FT magnitude is densely sampled in the frequency domain” [1]. Hayes et al. [2] developed a set of conditions for reconstruction (to within a scale factor) of a discrete-time signal by using only FT magnitude or phase. An iterative reconstruction algorithm was presented that utilized an M -point DFT. The authors reported the total squared error was observed to monotonically decrease with each iteration, as long as $M \geq 2N$, where N is the length of the discrete-time signal.

In this tutorial, we present a simple, noniterative procedure for reconstructing a discrete-time signal using only the RDFT. In contrast to the iterative methods that reconstructed a discrete-time signal using only FT magnitude or phase [1]–[3], this procedure can perfectly reconstruct the original signal, if appropriate zero-padding and DFT sizes are used. This implies that the RDFT captures all the information about the signal.

Signal reconstruction from the RDFT

RDFT reconstruction procedure

Let us assume an N -length discrete-time complex-valued signal $x(n)$, where $n = 0, 1, \dots, N - 1$. An extended signal $\tilde{x}(n)$ is created by first delaying $x(n)$ by one sample and padding a zero at the beginning and then padding a sufficient number of zeros at the end, to give it a length of $M \geq 2N + 1$

$$\tilde{x}(n) = \begin{cases} 0, & \text{for } n = 0 \\ x(n-1), & \text{for } n = 1, 2, \dots, N \\ 0, & \text{for } n = N+1, \dots, M-1 \end{cases} \quad (1)$$

[Zero-padding the first sample is only required if $x(n)$ is a complex-valued signal. If $x(n)$ is real-valued, the zero at the beginning is not necessary.]

An M -point discrete FT is performed on this extended signal

$$\tilde{X}(k) = \sum_{n=0}^{M-1} \tilde{x}(n) e^{-j2\pi nk/M}, \quad \text{for } k = 0, 1, \dots, M-1 \quad (2)$$

where k is the frequency sample. The complex DFT can be written as a sum of RI parts, $\tilde{X}_R(k)$ and $\tilde{X}_I(k)$, respectively,

$$\tilde{X}(k) = \tilde{X}_R(k) + j\tilde{X}_I(k). \quad (3)$$

To reconstruct the signal using only the real part $\tilde{X}_R(k)$, we first multiply it by a factor of two and then take the inverse DFT

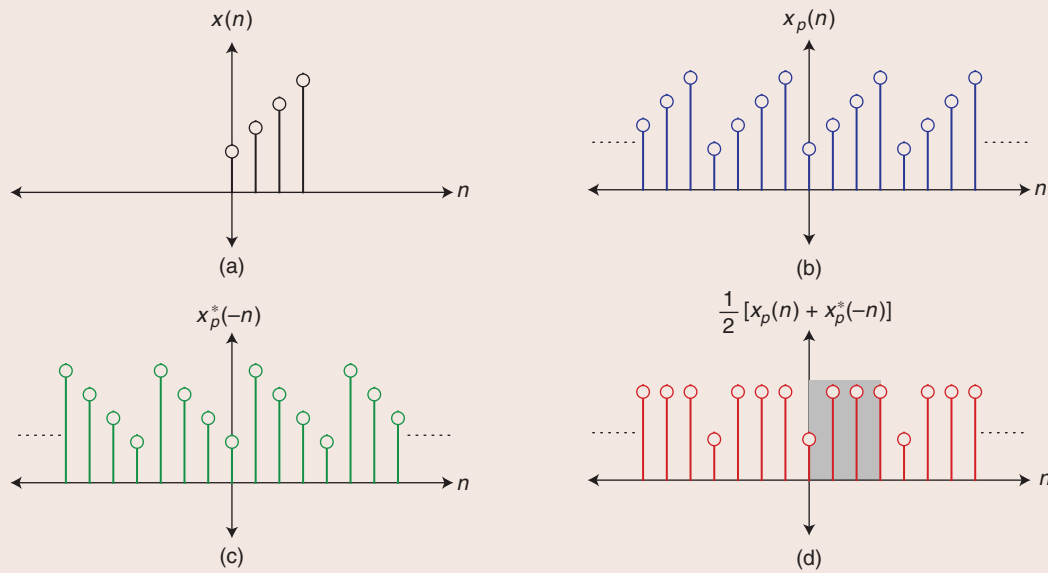


FIGURE 1. Plots show the reconstruction of a discrete-time signal (without zero-padding) from only the RDFT: (a) the original real-valued signal $x(n)$ with no delay or zero-padding ($N = 4$); (b) a periodic version of $x(n)$ (with a period of N) as a result of the cyclic properties of the DFT; (c) a complex conjugate and time-reversed version of $x_p(n)$; and (d) an output signal from taking the inverse DFT of the real part, showing the effect of time-domain aliasing. The shaded region shows the reconstructed signal.

$$\tilde{y}(n) = \frac{1}{M} \sum_{k=0}^{M-1} 2\tilde{X}_R(k) e^{j2\pi nk/M} \quad (4)$$

for $n = 0, 1, \dots, M-1$

The resulting signal $y(n)$ from the inverse DFT contains the original signal $x(n)$ embedded within it [see (5) in the box at the bottom of the page], where $x^*(n)$ is the complex conjugate of $x(n)$. Therefore, the original discrete-time signal $x(n)$ can be perfectly recovered from this procedure.

Theoretical basis for the RDFT

The theoretical basis for the reconstruction procedure that was described in the previous section comes from a DFT property from [4, eq. (8.110)] [see (6) in the box at the bottom of the page], where $x_p(n)$ is a periodic signal (with a period of N), where each period is identical to $x(n)$, and $\mathcal{R}\{X(k)\}$ is the RDFT of $x(n)$. When looking at this prop-

erty, recovering $x(n)$ from the RDFT alone does not appear possible without the imaginary part. A similar situation occurs in the corresponding property that relates the imaginary part of the DFT as follows [4]:

$$\frac{1}{2}[x_p(n) - x_p^*(-n)] \xrightarrow{\text{DFT}} j\mathcal{I}\{X(k)\}.$$

This is because of the presence of the complex conjugate term $x_p^*(-n)$, which completely overlaps with $x_p(n)$, and therefore results in time-domain aliasing. This is shown in Figure 1 for the case of a real-valued discrete-time signal of length $N = 4$. Figure 1(d) shows the final signal as a result of taking the inverse of the RDFT of $x(n)$ [shown in Figure 1(a)]. We can see the effects of aliasing distortion in Figure 1(d). Therefore, the original signal $x(n)$ cannot be perfectly recovered from the RDFT in this case.

To circumvent the problem of time-domain aliasing, the signal $x(n)$ is delayed by one sample and a zero is padded at the beginning to avoid the overlap at $n = 0$. Further padding of N zeros is performed at the end of the signal. This effectively extends the period of $x_p(n)$ from N to $2N + 1$. Therefore, by choosing the value of $M \geq 2N + 1$, we can prevent the time-domain aliasing. This is demonstrated in Figure 2 for a real-valued discrete-time signal of length $N = 4$. As we can see in Figure 2(d), the original signal $x(n)$ can be perfectly recovered from the RDFT if the appropriate delay and zero-padding are applied.

Discussion and applications

One potential application of the RDFT method of signal reconstruction is in the area of speech enhancement, where noise-corrupted speech is processed to alleviate the degrading effects of the noise and therefore improve the quality and intelligibility of the speech. Recent studies in this area [5], [6] have shown that, over time, the estimation of the RI DFT coefficients are as effective as modulation-magnitude domain processing, where the DFT magnitude coefficients are temporally processed and then combined with the noisy DFT

$$\tilde{y}(n) = \begin{cases} x(n-1), & \text{for } n=1, 2, \dots, N \\ x^*(M-1-n), & \text{for } n=M-N, \dots, M-1, \\ 0, & \text{elsewhere} \end{cases} \quad (5)$$

$$\frac{1}{2}[x_p(n) + x_p^*(-n)] \xrightarrow{\text{DFT}} \mathcal{R}\{X(k)\} \quad (6)$$

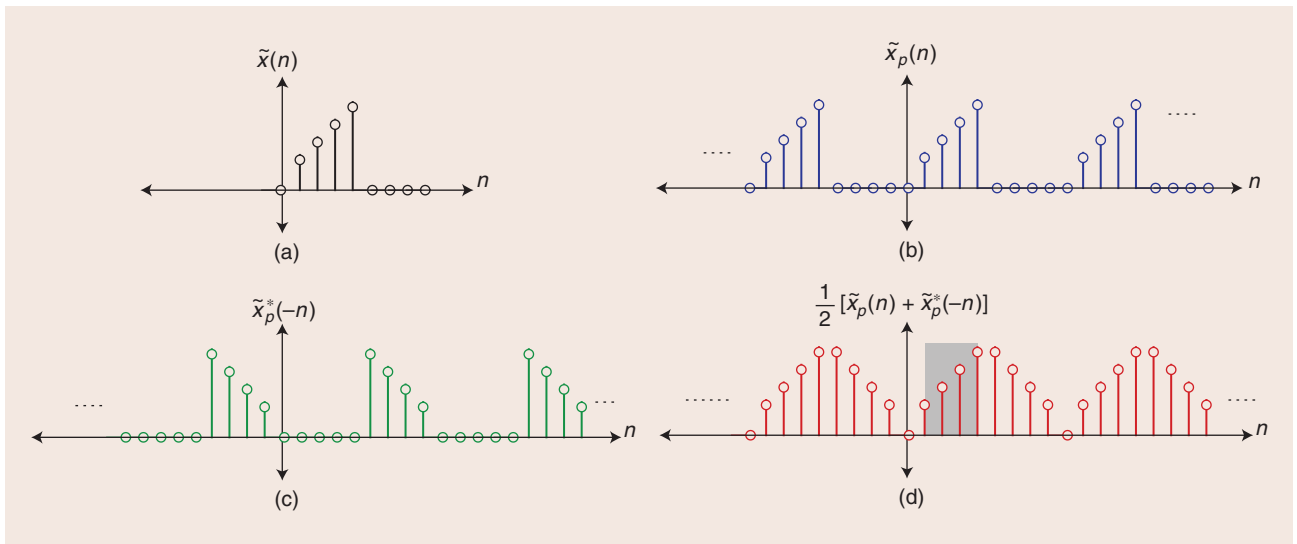


FIGURE 2. Plots showing the reconstruction of a discrete-time signal (with delay and zero-padding) from only the RDFT: (a) an extended real-valued signal $\tilde{x}(n)$ with delay and zero padding ($M = 2N + 1 = 9$); (b) a periodic version of $\tilde{x}(n)$ (with a period of M) as a result of the cyclic properties of the DFT; (c) a complex conjugate and time-reversed version of $\tilde{x}_p(n)$; and (d) an output signal from taking the inverse DFT of the real part, showing no time-domain aliasing. The shaded region contains the reconstructed signal.

phase in the synthesis stage. In particular, the enhanced speech from the application of minimum mean squared error estimation methods in the modulation-RI domain was found to have improved speech intelligibility [6]. In the RI approach to speech enhancement, the enhancement algorithm is applied independently on the RI parts, treating them as if they were time-domain signals. The theoretical advantages of modulation-RI processing include a valid additive-noise assumption in this domain as well as eliminating the step of combining with the noisy DFT phase in the synthesis stage [5].

First, there can be computational advantages of the RDFT method of signal reconstruction in the analysis-modification-synthesis (AMS) framework [7] that is used in the modulation-RI enhancement method, because the latter method requires the processing of two sets of signals (R and I) as opposed to the former method (R only). In the AMS framework, speech is windowed into overlapping short (e.g., 32 ms) frames and then a densely sampled DFT is computed for each frame. For speech signals sampled at 8 kHz, each frame would consist of 256 time-domain samples. In the case of a critically sampled M -point complex-valued DFT (where $M = 256$),

the computational complexity of the independent processing of RI parts would be identical to the RDFT (i.e., 257 real numbers), which itself requires a 513-point DFT. More specifically for this particular case, a 256-point complex-valued DFT would produce 256 real and 256 imaginary values. Since a real-valued signal is being considered, the complex-conjugation property of the DFT means that only 129 real (including the dc value) and 128 imaginary values need to be processed, which gives a total of 257.

For the RDFT processing, zero-padding is applied and a 513-point complex-valued DFT would produce 257 unique real values. However, a more densely sampled DFT is typically used in modulation-domain speech processing because of its finer spectral interpolation properties. In the densely sampled DFT case (e.g., $M = 512$), the RI processing requires 513 unique values to be processed, as opposed to 257 unique values for the RDFT case. This potentially represents a saving of roughly 50% in computational complexity.

Second, the reconstruction procedure suggests that the real (or imaginary) part of the DFT alone is sufficient for perfect signal reconstruction, as long as it is densely sampled in the frequency

axis; therefore, it captures all the information of the discrete-time signal $x(n)$. A similar procedure can be derived to perfectly reconstruct a signal from the imaginary part of the DFT using the same arguments. It can be inferred that after the DFT is performed, the signal information is replicated and then embedded among its RI parts. This raises a particularly interesting question of whether processing just the RDFT would be advantageous when compared with processing both RI parts, especially given the increasing interest in incorporating phase-related information in speech processing (such as [8]).

Another real-valued transform is the discrete cosine transform (DCT), which can be interpreted as the DFT of a symmetrically extended signal. We have compared the RDFT with the DCT in modulation domain speech enhancement experiments and will report the results in an upcoming paper.

Conclusions

In this tutorial, we have described a procedure for the reconstruction of a discrete-time, complex-valued signal from the RDFT. The RDFT procedure

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include ultrasound imaging, ultrasound elastography and thermography, and ultrasound system design. He is a staff systems engineer at Siemens Healthineers in Seattle, Washington.

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TIPS & TRICKS (continued from page 164)

can be used in applications that process real-valued signals in an analysis-modification-synthesis framework, where processing only the RDFT can provide a reduction in computational complexity for the case of a densely sampled DFT. In contrast to previous methods of signal reconstruction from partial DFT information (i.e., DFT magnitude or phase), the reconstructed signal from the procedure described in this tutorial has no errors when compared with the original signal.

Authors

Stephen So (s.so@griffith.edu.au) received his B.Eng. degree (first class honors) in microelectronic engineering and his Ph.D. degree in image and speech signal processing from Griffith University, Brisbane, Australia, in 1999 and 2005, respectively. Currently he is a senior lecturer in the Griffith School of Engineering, Gold Coast campus, Griffith University. He is a Senior Member of the IEEE and was a member of the editorial board of *Digital Signal Processing*

(Elsevier) from 2013 to 2015. His research interests are in image and speech coding, speech enhancement and noise reduction, distributed speech recognition, electrocardiography signal processing, machine learning, and deep neural networks.

Kuldip K. Paliwal (k.paliwal@griffith.edu.au) received his B.S. degree from Agra University, India, in 1969, his M.S. degree from Aligarh Muslim University, India, in 1971, and his Ph.D. degree from Bombay University, India, in 1978. Since July 1993, he has been a professor with Griffith University, Brisbane, Australia, in the Griffith School of Engineering. His current research interests include speech recognition, speech coding, speaker recognition, speech enhancement, face recognition, image coding, pattern recognition, and artificial neural networks. He has published more than 300 papers in these research areas. He is on the editorial board of *IEEE Signal Processing Magazine* and also served as an associate editor of *IEEE Signal Processing Letters* from 1997 to 2000. He has coedited two

books *Speech Coding and Synthesis* (Elsevier) and *Speech and Speaker Recognition: Advanced Topics* (Kluwer).

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